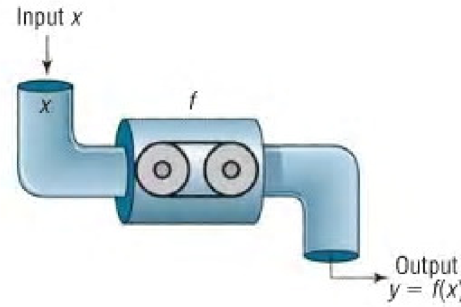
Functions

Relation: A correspondence between two sets, called the **domain** and the **range.**

Domain: The set of inputs (the x-values) of a relation.

Range: The set of outputs (the y-values) of a relation.

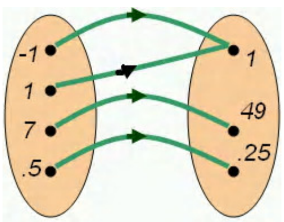
Function: A relation in which for each input there is *exactly one* output. Each element of the domain corresponds to exactly one element of the range.

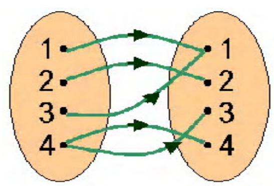


Function Machine Rules:

1. The machine only accepts inputs that are part of the domain.
2. The machine gets confused if there is more than one possible output for any one input. It only works if there is *only one output for each input*.

Examples: Express the relation shown in each map as a set of ordered pairs. Decide whether each relation is a function.

a) 

b) 

$\{(-1,1), (1,1), (7,49), (.5,.25)\}$
Function

$\{(1,1), (2,2), (3,1), (4,3), (4,4)\}$
Not a function - two possible outputs when input is 4.

Examples: For each relation, write the domain and range and determine whether the relation is a function.

a) $\{(1,4)(3,6)(5,8)(6,9)\}$ Domain: $\{1,3,5,6\}$ Range: $\{4,6,8,9\}$ Function

b) $\{(2,5)(2,6)(4,7)(6,9)\}$ Domain: $\{2,4,6\}$ Range: $\{5,6,7,9\}$ Not a function - two possible outputs when input is 2.

c) $\{(4,9)(6,7)(8,9)(10,11)\}$ Domain: $\{4,6,8,10\}$ Range: $\{7,9,11\}$ Function

Examples: Determine whether the equation defines y as a function of x.

a) $x^2 - y = 2$

$x^2 = y + 2$
$y = x^2 - 2$

yes. Each x gives only one y.

b) $x + y^2 = 1$

$y^2 = 1 - x$
$y = \pm\sqrt{1-x}$

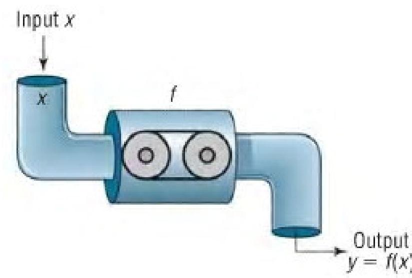
no. Some x's give more than one y.

c) $y = \dfrac{3x-1}{x+2}$

yes. Each x gives only one y.

Function Notation

We often use the letters f, g, and h to represent functions. The function machine to the right represents the function $y = f(x)$.



- f is the name of the function. It is the rule that relates x and y.
- x represents the input, called the ***independent variable*** or ***argument***.
- y or $f(x)$ represents the output, called the ***dependent variable*** (because the value of y *depends* on the value of x that is used as an input).

$f(x)$ is read "f of x," and means "the value (output) of the function f when the input is x."

$f(x)$ DOES NOT mean f times x!

Example: For the function $f(x) = -x + 9$, evaluate the following:

a) $f(2)$ $= -2 + 9$
$= \boxed{7}$

b) $f(x) + f(-6)$
$= -x + 9 + -(-6) + 9$
$= -x + 9 + 6 + 9$
$= \boxed{-x + 24}$

c) $3f(x)$
$= 3(-x + 9)$
$= \boxed{-3x + 27}$

d) $f(3x)$
$= -(3x) + 9$
$= \boxed{-3x + 9}$

e) $f(-x)$
$= -(-x) + 9$
$= \boxed{x + 9}$

f) $-f(x)$
$= -(-x + 9)$
$= \boxed{x - 9}$

g) $f(x) + 2$
$= -x + 9 + 2$
$= \boxed{-x + 11}$

h) $f(x + 2)$
$= -(x + 2) + 9$
$= -x - 2 + 9$
$= \boxed{-x + 7}$

i) $f(x + h)$
$= -(x + h) + 9$
$= \boxed{-x - h + 9}$

Example: For the function $f(x) = 3x^2 - 5x$, evaluate the following:

a) $f(2)$
$= 3(2)^2 - 5(2)$
$= 12 - 10$
$= \boxed{2}$

b) $f(x) + f(-6)$
$= 3x^2 - 5x + 3(-6)^2 - 5(-6)$
$= 3x^2 - 5x + 3(36) + 30$
$= \boxed{3x^2 - 5x + 138}$

c) $3f(x)$
$= 3(3x^2 - 5x)$
$= \boxed{9x^2 - 15x}$

d) $f(3x)$
$= 3(3x)^2 - 5(3x)$
$= 3(9x^2) - 15x$
$= \boxed{27x^2 - 15x}$

e) $f(-x)$
$= 3(-x)^2 - 5(-x)$
$= \boxed{3x^2 + 5x}$

f) $-f(x)$
$= -(3x^2 - 5x)$
$= \boxed{-3x^2 + 5x}$

g) $f(x) + 2$
$= \boxed{3x^2 - 5x + 2}$

h) $f(x + 2)$
$= 3(x + 2)^2 - 5(x + 2)$
$= 3(x^2 + 4x + 4) - 5x - 10$
$= 3x^2 + 12x + 12 - 5x - 10$
$= \boxed{3x^2 + 7x + 2}$

i) $f(x + h)$
$= 3(x + h)^2 - 5(x + h)$
$= 3(x^2 + 2xh + h^2) - 5x - 5h$
$= \boxed{3x^2 + 6xh + 3h^2 - 5x - 5h}$

The Difference Quotient

The expression $\dfrac{f(x+h)-f(x)}{h}$, $h \neq 0$ is called the ***difference quotient.*** It is the basis of many of the ideas in calculus. One thing that makes finding the difference quotient of a function easier is to look at all the parts of the expression separately: $f(x+h)$, $-f(x)$, and h.

Examples: Find the difference quotient of f; that is, find $\dfrac{f(x+h)-f(x)}{h}$, $h \neq 0$. Be sure to simplify.

a) $f(x) = 2x - 5$

$f(x+h) = 2(x+h) - 5$

$-f(x) = -(2x-5)$

$$\frac{2(x+h)-5-(2x-5)}{h} = \frac{\cancel{2x}+2h\cancel{-5}\cancel{-2x}\cancel{+5}}{h} = \frac{2h}{h} = \boxed{2}$$

b) $f(x) = 4x^2 + x$

$f(x+h) = 4(x+h)^2 + (x+h)$

$-f(x) = -(4x^2+x)$

$$\frac{4(x+h)^2+(x+h)-(4x^2+x)}{h} = \frac{4(x^2+2xh+h^2)+x+h-4x^2-x}{h} = \frac{\cancel{4x^2}+8xh+4h^2\cancel{+x}+h\cancel{-4x^2}\cancel{-x}}{h} = \frac{8xh+4h^2+h}{h} = \boxed{8x+4h+1}$$

c) $f(x) = 5x^2 - x + 4$

$f(x+h) = 5(x+h)^2 - (x+h) + 4$

$-f(x) = -(5x^2 - x + 4)$

$$\frac{5(x+h)^2-(x+h)+4-(5x^2-x+4)}{h} = \frac{5(x^2+2xh+h^2)-x-h+4-5x^2+x-4}{h} = \frac{\cancel{5x^2}+10xh+5h^2\cancel{-x}-h\cancel{+4}\cancel{-5x^2}\cancel{+x}\cancel{-4}}{h} = \frac{10xh+5h^2-h}{h} = \boxed{10x+5h-1}$$

d) $f(x) = \dfrac{1}{x-6}$

$f(x+h) = \dfrac{1}{(x+h)-6}$

$-f(x) = -\dfrac{1}{x-6}$

$$\frac{\frac{1}{x+h-6}+\frac{-1}{x-6}}{h} = \frac{\frac{x-6}{(x+h-6)(x-6)}+\frac{-1(x+h-6)}{(x+h-6)(x-6)}}{h} = \frac{\frac{\cancel{x}\cancel{-6}\cancel{-x}-h\cancel{+6}}{(x+h-6)(x-6)}}{h} = \frac{-h}{(x+h-6)(x-6)}\cdot\frac{1}{h} = \boxed{\frac{-1}{(x+h-6)(x-6)}}$$

Implicit and Explicit Forms of a Function

When an equation has not been solved for y, we say that the function is written ***implicitly.*** If it is possible to solve the equation for y in terms of x, then we write $y = f(x)$ and say that the function is given ***explicitly.*** In order to enter an equation into a graphing calculator, it must be written in explicit form.

Examples: Write the explicit form of each function.

a) $6x + 3y = 9$

$$\frac{3y}{3} = \frac{-6x+9}{3}$$

$$\boxed{y = f(x) = -2x+3}$$

b) $x^2 - y = 3$

$$x^2 = y + 3$$

$$\boxed{y = f(x) = x^2 - 3}$$

c) $\dfrac{xy}{x} = \dfrac{10}{x}$

$$\boxed{y = f(x) = \frac{10}{x}}$$

Domain of a Function

The domain of a function $f(x)$ is the set of all inputs *x*.

- If the function is listed in a table or as a set of ordered pairs, the domain is the set of all first coordinates.
- If the function is described by a graph, the domain is the set of all *x*-coordinates of the points on the graph.
- If the function is described by an equation, the domain is the set of all real numbers for which $f(x)$ is a real number. Figure out if there are any *x*-values that cause "problems" (zero in a denominator, square root of a negative, etc.) when you plug them into the function. If so, these numbers are not part of the domain.
- If the function is used in an application, the domain is the set of all numbers that make sense in the problem.

Tips for finding domain:

1. If the equation has fractions, exclude any numbers that give a zero in a denominator.
2. If the equation has an even root, exclude any numbers that cause the expression under the root to be negative.

Examples: Determine the domain of $f(x)$.

a) $f(x) = x^2 - x$

$\boxed{\mathbb{R} \text{ or } (-\infty, \infty)}$

b) $f(x) = \dfrac{4x}{x^2 - 9}$

$x^2 - 9 \neq 0$

$(x+3)(x-3) \neq 0$

$\boxed{\{x \mid x \neq -3, 3\} \text{ or } (-\infty,-3)\cup(-3,3)\cup(3,\infty)}$

c) $f(x) = |x - 5|$

$\boxed{\mathbb{R} \text{ or } (-\infty, \infty)}$

d) $f(x) = \sqrt{x+2}$

$x + 2 \geq 0$

$\boxed{\{x \mid x \geq -2\} \text{ or } [-2, \infty)}$

e) $f(x) = \dfrac{1}{\sqrt{x+2}}$

$x + 2 \geq 0$ & $\sqrt{x+2} \neq 0$

$x + 2 > 0$

$\boxed{\{x \mid x > -2\} \text{ or } (-2, \infty)}$

f) $f(x) = \sqrt{-5x + 7}$

$-5x + 7 \geq 0$

$\dfrac{-5x}{-5} \geq \dfrac{-7}{-5}$

$x \leq \frac{7}{5}$

$\boxed{\{x \mid x \leq \tfrac{7}{5}\} \text{ or } (-\infty, 7/5]}$

Divided by a negative. Flip inequality sign!

Example: Express the area of a circle as a function of its radius. Find the domain of the function.

$\boxed{A(r) = \pi r^2 \quad \{r \mid r > 0\}}$ ← circles with negative or zero radius don't make sense.

Sums, Differences, Products, and Quotients of Two Functions

The **sum** $f + g$ is defined by $(f+g)(x) = f(x) + g(x)$

The **difference** $f - g$ is defined by $(f-g)(x) = f(x) - g(x)$

The **product** $f \cdot g$ is defined by $(f \cdot g)(x) = f(x) \cdot g(x)$

The **quotient** $\dfrac{f}{g}$ is defined by $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)},\ g(x) \neq 0$

The domain of $f+g$, $f-g$, or $f \cdot g$ consists of all the numbers x that are in the domains of both f and g. The domain of f/g consists of all the numbers x for which $g(x) \neq 0$ that are in the domains of both f and g.

Examples: Find the following and determine the domain given the functions $f(x) = \dfrac{x}{x+3}$ and $g(x) = \dfrac{x+8}{x+3}$.

Domain: $\{x \mid x \neq -3\}$ Domain: $\{x \mid x \neq -3\}$

a) $(f+g)(x)$

$= \dfrac{x}{x+3} + \dfrac{x+8}{x+3}$

$= \boxed{\dfrac{2x+8}{x+3}}$

$\boxed{D: \{x \mid x \neq -3\}}$

b) $(f-g)(x)$

$= \dfrac{x}{x+3} - \dfrac{x+8}{x+3}$

$= \dfrac{x - x - 8}{x+3}$

$= \boxed{\dfrac{-8}{x+3}}$

$\boxed{D: \{x \mid x \neq -3\}}$

c) $(f \cdot g)(x)$

$= \dfrac{x}{x+3} \cdot \dfrac{x+8}{x+3}$

$= \boxed{\dfrac{x(x+8)}{(x+3)^2}}$

$\boxed{D: \{x \mid x \neq -3\}}$

d) $\left(\dfrac{f}{g}\right)(x)$

$= \dfrac{\frac{x}{x+3}}{\frac{x+8}{x+3}}$

This can't equal zero

$\dfrac{x+8}{x+3} \neq 0$

$x + 8 \neq 0$

$x \neq -8$

$= \dfrac{x}{\cancel{x+3}} \cdot \dfrac{\cancel{x+3}}{x+8}$

$= \boxed{\dfrac{x}{x+8}}$

$\boxed{D: \{x \mid x \neq -3, -8\}}$